The Finite Quasi-equational Base Problem

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All algebras are of a finite (fixed) type.

Definition. A nontrivial class $\mathcal{Q}$ of algebras is a *quasi-variety* if it is closed under I (taking isomorphic algebras), S (subalgebras), P (products) and P_U (ultraproducts).

Definition. The *quasi-variety* $\mathcal{Q}(\mathbf{A})$ *generated by an algebra* $\mathbf{A}$ is the smallest quasi-variety containing $\mathbf{A}$.

Definition. A *quasi-identity* is a formula of the form $(p_1 \approx q_1 \ \& \ \ldots \ \& \ p_n \approx q_n) \rightarrow p \approx q$.

Theorem. *A class of algebras* $\mathcal{Q}$ *is a* quasi-variety *if and only if* $\mathcal{Q}$ *can be axiomatized by quasi-identities.*

Definition. An algebra $\mathbf{A}$ is *finitely q-based* if $\mathcal{Q}(\mathbf{A})$ can be finitely axiomatized (by quasi-identities). $\mathbf{A}$ is *inherently nonfinitely q-based* if there is no finitely axiomatizable locally finite quasi-variety containing $\mathbf{A}$.

Problem (The finite quasi-equational base problem)**.** Is it decidable for a finite algebra if it is finitely q-based?

Definition. An algebra $\mathbf{A}$ is a *semilattice algebra* if its signature contains (among other symbols) a binary symbol $\wedge$ (*the meet*) such that

(1) $\langle A; \wedge \rangle$ is a semilattice.

$\mathbf{A}$ is said to be *compatible* if it satisfies the equations

(2) $f(z_1, \ldots, z_{i-1}, x \wedge y, z_{i+1}, \ldots, z_n) \approx f(\ldots, x, \ldots) \wedge f(\ldots, y, \ldots)$ for every n-ary operation f of σ and every $i \in \{1, \ldots, n\}$.

Example. The following are compatible semilattice algebras:

- digraph algebras
- semilattices with a finite set of endomorphisms
- flat algebras over any quasigroup

Definition. For a variable x, *basic x-terms* of depth n are defined as follows. The term x is the only basic x-term of depth 0. For $n > 0$, basic x-terms of depth n are the terms $f(z_1, \dots, z_{i-1}, t, z_{i+1}, \dots, z_n)$ such that f is an n-ary basic operation, $1 \leq i \leq n$, t is a basic x-term of depth $n-1$ and $z_1, \dots, z_n$ are variables different from x.

A *basic polynomial of* $\mathbf{A}$ is a unary polynomial $p(x) = t(x; a_1, a_2, \dots)$ where $t(x; z_1, z_2, \dots)$ is a basic x-term and $a_1, a_2, \dots \in A$.

Fact. *A semilattice algebra $\mathbf{A}$ is compatible if and only if $p(a \wedge b) = p(a) \wedge p(b)$ for all basic polynomials p of $\mathbf{A}$ and all elements $a, b \in A$.*

Fact. *Let $\mathbf{A}$ be a compatible semilattice algebra and F be a filter of $\mathbf{A}$. Then for every basic polynomial p of $\mathbf{A}$, $p^{-1}(F)$ is either empty or a filter of $\mathbf{A}$.*

Lemma. *Let* $\mathbf{A}$ *be a compatible semilattice algebra and* F *be a filter of* $\mathbf{A}$*. Put*

$$\mathcal{C}_F = \{\, p^{-1}(F) : p \textit{ is a basic polynomial of } \mathbf{A} \,\}$$

$$\vartheta_F = \bigcap \{\, H^2 \cup (A \setminus H)^2 : H \in \mathcal{C}_F \,\}.$$

Then ϑ_F *is a congruence of* $\mathbf{A}$*.*

Lemma. *Let* $\mathbf{A}$ *be a compatible semilattice algebra,* F *be a principal filter generated by a join irreducible element* $d \in A$*, and* $c \in A$ *be the unique lower cover of* d*. Then* ϑ_F *is the largest congruence that does not collapse* c *and* d*;* ϑ_F *and* $\operatorname{Cg}_{\mathbf{A}}(c, d)$ *form a splitting pair of congruences in* $\operatorname{Con} \mathbf{A}$*.*

We fix a finite compatible semilattice algebra $\mathbf{A}$. Put $K = |A|$.

Definition. Denote by $\mathcal{Q}_1$ the quasi-variety determined by the equations (1) and (2) and for every $\binom{K+1}{2}$-tuple of basic x-terms $t_{1,2}, t_{1,3}, \ldots, t_{K,K+1}$ of depth $\leq K+1$, and every $\binom{K+1}{2}$-tuple $\varepsilon_{1,2}, \varepsilon_{1,3}, \ldots, \varepsilon_{K,K+1} \in \{0,1\}$ the following quasi-equation $\gamma_{\bar{t},\bar{\varepsilon}}$

(3) $(x \leq y \mathbin{\&} D_{1,2} \mathbin{\&} D_{1,3} \mathbin{\&} \ldots \mathbin{\&} D_{K,K+1}) \to x \approx y$

where

$$D_{i,j} = \begin{cases} t_{i,j}(u_i) \geq y \mathbin{\&} t_{i,j}(u_j) \wedge y \leq x & \text{if } \varepsilon_{i,j} = 0, \\ t_{i,j}(u_j) \geq y \mathbin{\&} t_{i,j}(u_i) \wedge y \leq x & \text{if } \varepsilon_{i,j} = 1. \end{cases}$$

Lemma. *The quasi-variety $\mathcal{Q}_1$ is finitely axiomatized and contains* $\mathbf{A}$.

Lemma. *Let* $\mathbf{B} \in \mathcal{Q}_1$, $a, b \in B$ *two elements such that* $b \not\leq a$, *and let* F *be a maximal filter of* $\mathbf{B}$ *such that* $b \in F$ *and* $a \notin F$. *Then* $\mathcal{C}_F = \{p_1^{-1}(F), \ldots, p_r^{-1}(F)\}$ *for some* $r \leq K$ *and basic polynomials* $p_1, \ldots, p_r$ *of* $\mathbf{B}$ *of depth* $\leq K$. *Moreover,* $|B/\vartheta_F| \leq K$.

Corollary. $\mathcal{Q}_1$ *is locally finite; every algebra of* $\mathcal{Q}_1$ *is a subdirect product of algebras of size* $\leq K$*. Consequently,* $\mathbf{A}$ *is not inherently nonfinitely q-based.*

Definition. Denote by $\mathcal{Q}_2$ the quasi-variety determined by the quasi-equations $(1) - (3)$ and all quasi-equations in at most K variables that are satisfied in $\mathbf{A}$.

Theorem. *Let $\mathbf{B} \in \mathcal{Q}_2$, $a, b \in B$ two elements such that $b \not\leq a$, and let F be a maximal filter of $\mathbf{B}$ such that $b \in F$ and $a \notin F$. Then $\mathbf{B}/\vartheta_F \cong \mathbf{C}/\vartheta_H$ where $\mathbf{C}$ is a subalgebra of $\mathbf{A}$ and H is a principal filter generated by a join irreducible element of $\mathbf{C}$.*

Corollary. *Let* $\mathbf{A}$ *be a finite compatible semilattice algebra such that* $HS(\mathbf{A}) \subseteq \mathcal{Q}(\mathbf{A})$. *Then* $\mathbf{A}$ *is finitely q-based.*

Corollary. *Every finite digraph algebra is finitely q-based.*

Corollary. *The flat algebra over any finite quasigroup (considered as a groupoid) is finitely q-based.*

Corollary. *Let* $\mathbf{A} = \langle A; \wedge, 0, f \rangle$ *be a finite flat compatible semilattice algebra with a unary operation* f. *Then* $\mathbf{A}$ *is finitely q-based.*

Theorem. *Let σ be a finite signature containing, in addition to $\wedge$ and 0, at least two unary symbols f and g (and, possibly, some other operation symbols). Then every finite compatible flat σ-algebra can be embedded into two finite compatible flat σ-algebras, one finitely q-based and the other one not finitely q-based.*

Problem. Is it decidable for a finite compatible semilattice algebra if it is finitely q-based?

Problem. Find the analog of the residual character of $\mathcal{V}(\mathbf{A})$ for the quasi-variety $\mathcal{Q}(\mathbf{A})$ which (under some restrictions) would imply that $\mathbf{A}$ is or is not finitely q-based.

Conjecture. Let $\mathbf{A}$ be a finite compatible semilattice algebra, and N be a positive integer. Then there exists a finite set Γ of quasi-equations such that for all semilattice algebras $\mathbf{B}$ of depth $\leq N$, $\mathbf{B} \in \mathcal{Q}(\mathbf{A})$ if and only if $\mathbf{B} \models \Gamma$.